Solving systems of linear equations using matrices

A *Maplet* for elementary row operations

A. G. Martin

University of Kent

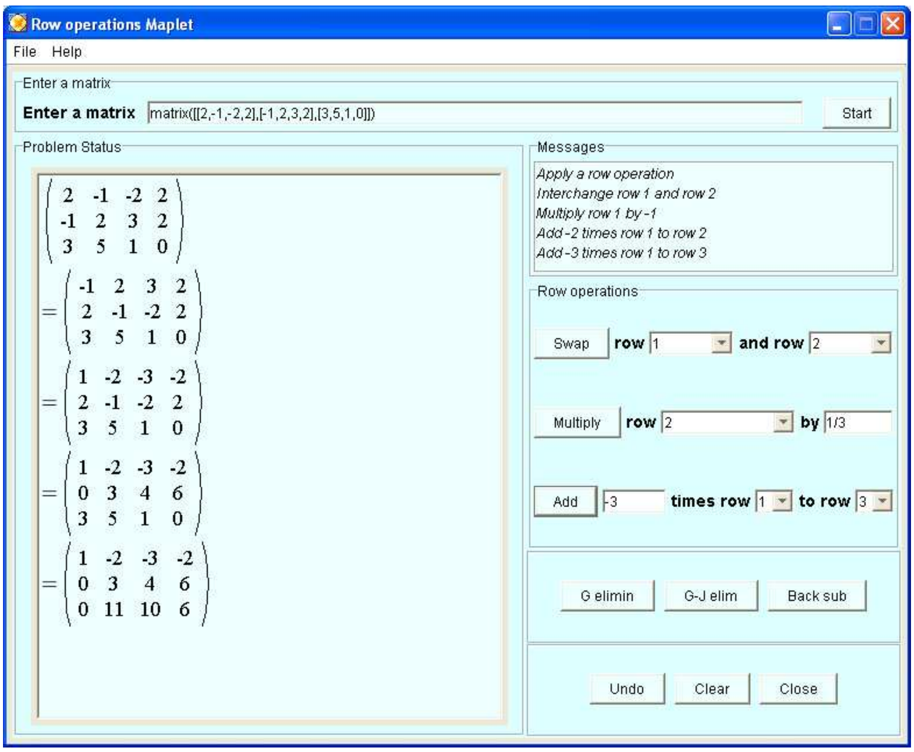
agmweb@ilovemaths.co.uk

http://www.ilovemaths.co.uk

August 24, 2003

Abstract

A *Maplet* is a custom graphical user interface designed to access *Maple*'s computational engine. The *Maplet* presented accesses *Maple*'s elementary row operation functions. The *Maplet* and accompanying student handout can be downloaded from the author's *Maple* page, http://www.ilovemaths.co.uk/maple/.



Solving systems of linear equations using matrices

Solve the linear system of equations;

$$\begin{aligned} 2x - y - 2z &= 2 \\ -x + 2y + 3z &= 2 \\ 3x + 5y + z &= 0 \end{aligned}$$

The **augmented matrix** of the system is

$$\left[\begin{array}{rrr|r} 2 & -1 & -2 & 2 \\ -1 & 2 & 3 & 2 \\ 3 & 5 & 1 & 0 \end{array}\right]$$

where the left hand side is called the **coefficient matrix** of the system.

Your task is to use elementary row operations to reduce the augmented matrix to **echelon (or triangular) form**. This is called **Gaussian elimination**.

Recall that echelon form looks like

$$\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & * & * \end{array}\right]$$

The first step is obtaining a 1 as the upper left entry. You can achieve this by first swapping rows 1 and 2, followed by multiplying row 1 by -1. At this stage your augmented matrix should look like,

$$\left[\begin{array}{rrr|r} 1 & -2 & -3 & -2 \\ 2 & -1 & -2 & 2 \\ 3 & 5 & 1 & 0 \end{array}\right]$$

This gives you a 1 in the first column which can be used to obtain zeros in the first column of the rows below the 1. To obtain 0 in the first column of the second row, subtract two times row 1 from row 2.

Continue with row operations until you obtain echelon form. If you wish to undo steps, press the "Undo" button.

An m-by-n (m rows, n columns) matrix may be transformed to echelon form by a sequence of at most mn elementary row operations. For this example you should therefore have obtained the matrix

$$\left[\begin{array}{rrr|r} 1 & -2 & -3 & -2 \\ 0 & 1 & 4/3 & 2 \\ 0 & 0 & -14/3 & -16 \end{array}\right]$$

in no more than 12 elementary row operations. (Echelon form can be achieved by 6 elementary row operations.) Can the sequence of row operations you used be improved?

The augmented matrix has been transformed into triangular form and the system of equations can be solved by back substituting. Solving the third equation, $-14/3z = -16$ gives $z = 24/7$ and back-substituting in the second equation gives $y = -18/7$, with the first equation giving $x = 22/7$. You can check this by pressing the "Back sub" button.

Now we complete the elimination to obtain **row reduced echelon form** by continuing with row operations until the coefficient matrix is the identity matrix. This is called **Jordon (or Gauss-Jordon) elimination**.

When the augmented matrix is in row reduced echelon form it looks like

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$$

Starting with echelon form,

$$\left[\begin{array}{ccc|c} 1 & -2 & -3 & -2 \\ 0 & 1 & 4/3 & 2 \\ 0 & 0 & -14/3 & -16 \end{array}\right]$$

you can start by using the 1 in the second row to obtain 0 in place of -2 in the first row. To do this, multiply row 2 by 2 and add to row 1. (If you pressed "Back sub" earlier you can undo it by pressing "Undo".) Continue with row operations until you obtain the row reduced echelon form for the augmented matrix,

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 22/7 \\ 0 & 1 & 0 & -18/7 \\ 0 & 0 & 1 & 24/7 \end{array}\right]$$

The solution of the system can be read off from the augmented matrix by inspection.

Exercises

Choose some standard exercises for here

Extension

Use the *Maplet* to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$$

Hint: Form the augmented matrix with the identity matrix and reduce to row reduced echelon form. i.e. transform the augmented matrix $[\ A \mid I\ ]$ to $[\ I \mid A^{-1}\ ]$ using elementary row operations.

Find the inverse of the 4×4 matrix, B.

$$B = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 3 & 0 & 0 & 1 \\ -1 & 1 & -2 & 1 \\ 4 & -1 & 1 & 0 \end{bmatrix}$$

Try to find the inverse of

$$C = \begin{bmatrix} 2 & -1 & 1 \\ 4 & 7 & 1 \\ 1 & 4 & 0 \end{bmatrix}$$

What problem do you face?
What does this tell you about C^{-1}?

A. G. Martin.
August 24, 2003